

# EXTENSION THEOREMS OF SAKAI TYPE FOR SEPARATELY HOLOMORPHIC AND MEROMORPHIC FUNCTIONS

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**ABSTRACT.** We first exhibit counterexamples to some open questions related to a theorem of Sakai. Then we establish an extension theorem of Sakai type for separately holomorphic/meromorphic functions.

## 1. INTRODUCTION

We first fix some notations and terminology. Throughout the paper,  $E$  denotes the unit disc of  $\mathbb{C}$ , and for any set  $S \subset \mathbb{C}^n$ ,  $\text{int } S$  (or equivalently  $\text{int}_{\mathbb{C}^n} S$ ) denotes the interior of  $S$ . For any domain  $D \subset \mathbb{C}^n$ , we say that the subset  $S \subset D$  does not separate domains in  $D$  if for every domain  $U \subset D$ , the set  $U \setminus S$  is connected. Moreover  $\mathcal{O}(D)$  (resp.  $\mathcal{M}(D)$ ) will denote the space of holomorphic (resp. meromorphic) functions on  $D$ . Finally, if  $S$  is a subset of  $D \times G$ , where  $D \subset \mathbb{C}^p$ ,  $G \subset \mathbb{C}^q$  are some open sets, then for  $a \in D$  (resp.  $b \in G$ ), the fiber  $S(a, \cdot)$  (resp.  $S(\cdot, b)$ ) is the set  $\{w \in G : (a, w) \in S\}$  (resp.  $\{z \in D : (z, b) \in S\}$ ).

In 1957 E. Sakai [9] claimed that he had proved the following result

**Theorem.** *Let  $S \subset E \times E$  be a relatively closed set such that  $\text{int } S = \emptyset$  and  $S$  does not separate domains in  $E \times E$ . Let  $A$  (resp.  $B$ ) be the set of all  $a \in E$  (resp.  $b \in E$ ) such that  $\text{int}_{\mathbb{C}} S(a, \cdot) = \emptyset$  (resp.  $\text{int}_{\mathbb{C}} S(\cdot, b) = \emptyset$ ). Put  $X := \mathbb{X}(A, B; E, E) = (A \times E) \cup (E \times B)$ .*

*Then for every function  $f : X \setminus S \rightarrow \mathbb{C}$  which is separately meromorphic on  $X$ , there exists an  $\hat{f} \in \mathcal{M}(E \times E)$  such that  $\hat{f} = f$  on  $X \setminus S$ .*

Unfortunately, it turns out as reported in [4] that the proof of E. Sakai contains an essential gap. In the latter paper M. Jarnicki and the first author also give a correct proof of this theorem.

E. Sakai also claimed in [9] that the following question (the  $n$ -dimensional version of the Theorem) can be answered positively but he did not give any proof.

**Question 1.** *For any  $n \geq 3$ , let  $S \subset E^n$  be a relatively closed set such that  $\text{int } S = \emptyset$  and  $S$  does not separate domains. Let  $f : E^n \setminus S \rightarrow \mathbb{C}$  be such that for any  $j \in \{1, \dots, n\}$  and for any  $(a', a'') \in E^{j-1} \times E^{n-j}$ , for which  $\text{int}_{\mathbb{C}} S(a', \cdot, a'') = \emptyset$ , the function  $f(a', \cdot, a'')$  extends meromorphically to  $E$ . Does  $f$  always extend meromorphically to  $E^n$ ?*

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In connection with the Theorem and Question 1, M. Jarnicki and the first author [4] posed two more questions :

**Question 2.** *Let  $\mathcal{A}$  be a subset of  $E^n$  ( $n \geq 2$ ) which is plurithin at  $0 \in E^n$  (see Section 2 below for the notion "plurithin"). For an arbitrary open neighborhood  $U$  of 0, does there exist a non-empty relatively open subset  $C$  of a real hypersurface in  $U$  such that  $C \subset U \setminus \mathcal{A}$  ?*

**Question 3.** *Let  $D \subset \mathbb{C}^p$ ,  $G \subset \mathbb{C}^q$  ( $p, q \geq 2$ ) be pseudoconvex domains and let  $S \subset D \times G$  be a relatively closed set such that  $\text{int } S = \emptyset$  and  $S$  does not separate domains in  $D \times G$ . Let  $A$  (resp.  $B$ ) be the set of all  $a \in D$  (resp.  $b \in G$ ) such that  $\text{int}_{\mathbb{C}^p} S(a, \cdot) = \emptyset$  (resp.  $\text{int}_{\mathbb{C}^q} S(\cdot, b) = \emptyset$ ). Put  $X := \mathbb{X}(A, B; D, G) = (A \times G) \cup (D \times B)$  and let  $f : X \setminus S \rightarrow \mathbb{C}$  be a function which is separately meromorphic on  $X$ . Does there always exist a function  $\hat{f} \in \mathcal{M}(D \times G)$  such that  $\hat{f} = f$  on  $X \setminus S$  ?*

This Note has two purposes. The first one is to give counterexamples to the three open questions above. The second one is to describe the maximal domain to which the function  $f$  in Questions 1 and 3 can be meromorphically extended.

This paper is organized as follows.

We begin Section 2 by collecting some background of the pluripotential theory and introducing some notations. This preparatory is necessary for us to state the results afterwards.

Section 3 provides three counterexamples to the three open questions from above.

The subsequent sections are devoted to the proof of a result in the positive direction. More precisely, we describe qualitatively the maximal domain of meromorphic extension of the function  $f$  in Questions 1 and 3. Section 4 develops auxiliary tools that will be used in Section 5 to prove the positive result.

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## 2. BACKGROUND AND STATEMENT OF THE RESULTS

We keep the main notation from [4].

Let  $N \in \mathbb{N}$ ,  $N \geq 2$ , and let  $\emptyset \neq A_j \subset D_j \subset \mathbb{C}^{n_j}$ , where  $D_j$  is a domain,  $j = 1, \dots, N$ . We define an  $N$ -fold cross

$$\begin{aligned} X &= \mathbb{X}(A_1, \dots, A_N; D_1, \dots, D_N) \\ &:= \bigcup_{j=1}^N A_1 \times \dots \times A_{j-1} \times D_j \times A_{j+1} \times \dots \times A_N \subset \mathbb{C}^{n_1 + \dots + n_N} = \mathbb{C}^n. \end{aligned}$$

For an open set  $\Omega \subset \mathbb{C}^n$  and  $A \subset \Omega$ , put

$$h_{A, \Omega} := \sup \{ u : u \in \mathcal{PSH}(\Omega), u \leq 1 \text{ on } \Omega, u \leq 0 \text{ on } A \},$$

where  $\mathcal{PSH}(\Omega)$  denotes the set of all plurisubharmonic functions on  $\Omega$ . Put

$$\omega_{A, \Omega} := \lim_{k \rightarrow +\infty} h_{A \cap \Omega_k, \Omega_k}^*,$$

where  $\{\Omega_k\}_{k=1}^\infty$  is a sequence of relatively compact open sets  $\Omega_k \subset \Omega_{k+1} \Subset \Omega$  with  $\bigcup_{k=1}^\infty \Omega_k = \Omega$  ( $h^*$  denotes the upper semicontinuous regularization of  $h$ ). We say that a subset  $\emptyset \neq A \subset \mathbb{C}^n$  is *locally pluriregular* if  $h_{A \cap \Omega, \Omega}^*(a) = 0$  for any  $a \in A$  and for any open neighborhood  $\Omega$  of  $a$ . We say that  $A$  is *plurithin* at a point  $a \in \mathbb{C}^n$  if either  $a \notin \overline{A}$  or  $a \in \overline{A}$  and  $\limsup_{A \setminus \{a\} \ni z \rightarrow a} u(z) < u(a)$  for a suitable function  $u$  plurisubharmonic in a neighborhood of  $a$ . For a good background of the pluripotential theory, see the books [5] or [1].

For an  $N$ -fold cross  $X := \mathbb{X}(A_1, \dots, A_N; D_1, \dots, D_N)$  let

$$\widehat{X} := \left\{ (z_1, \dots, z_N) \in D_1 \times \dots \times D_N : \sum_{j=1}^N \omega_{A_j, D_j}(z_j) < 1 \right\}.$$

Suppose that  $S_j \subset (A_1 \times \dots \times A_{j-1}) \times (A_{j+1} \times \dots \times A_N)$ ,  $j = 1, \dots, N$ . Define the *generalized  $N$ -fold cross*

$$\begin{aligned} T &= \mathbb{T}(A_1, \dots, A_N; D_1, \dots, D_N; S_1, \dots, S_N) := \bigcup_{j=1}^N \left\{ (z', z_j, z'') \right. \\ &\quad \left. \in (A_1 \times \dots \times A_{j-1}) \times D_j \times (A_{j+1} \times \dots \times A_N) : (z', z'') \notin S_j \right\}. \end{aligned}$$

Let  $M \subset T$  be a relatively closed set. We say that a function  $f : T \setminus M \rightarrow \mathbb{C}$  (resp.  $f : (T \setminus M) \setminus S \rightarrow \mathbb{C}$ ) is *separately holomorphic* and write  $f \in \mathcal{O}_s(T \setminus M)$  (resp. *separately meromorphic* and write  $f \in \mathcal{M}_s(T \setminus M)$ ) if for any  $j \in \{1, \dots, N\}$  and  $(a', a'') \in (A_1 \times \dots \times A_{j-1}) \times (A_{j+1} \times \dots \times A_N) \setminus S_j$  the function  $f(a', \cdot, a'')$  is holomorphic on (resp. can be meromorphically extended to) the open set  $D_j \setminus M(a', \cdot, a'')$ , where  $M(a', \cdot, a'') := \{z_j \in \mathbb{C}^{n_j} : (a', z_j, a'') \in M\}$ .

We are now ready to state the results. The following propositions give negative answers to Questions 2, 3 and 1 respectively.

**Proposition A.** *For any  $n \geq 2$ , there is an open dense subset  $\mathcal{A}$  of  $E^n$  which is plurithin at 0 and there exists no non-empty relatively open subset  $C$  of a real hypersurface such that  $C \subset E^n \setminus \mathcal{A}$ .*

**Proposition B.** *Let  $D \subset \mathbb{C}^p$ ,  $G \subset \mathbb{C}^q$  ( $p, q \geq 2$ ) be pseudoconvex domains. Then there is a relatively closed set  $S \subset D \times G$  with the following properties*

- (i)  $\text{int } S = \emptyset$  and  $S$  does not separate domains;
- (ii) let  $A$  (resp.  $B$ ) be the set of all  $a \in D$  (resp.  $b \in G$ ) such that  $\text{int}_{\mathbb{C}^p} S(a, \cdot) = \emptyset$  (resp.  $\text{int}_{\mathbb{C}^q} S(\cdot, b) = \emptyset$ ) and put  $X := \mathbb{X}(A, B; D, G)$ , then there exists a function  $f : X \setminus S \rightarrow \mathbb{C}$  which is separately holomorphic on  $X$  and there is no function  $\hat{f} \in \mathcal{M}(D \times G)$  such that  $\hat{f} = f$  on  $X \setminus S$ .

**Proposition C.** *For all  $n \geq 3$ , there is a relatively closed set  $S \subset E^n$  with the following properties*

- (i)  $\text{int } S = \emptyset$  and  $S$  does not separate domains;
- (ii) for  $1 \leq j \leq n$ , let  $S_j$  denote the set of all  $(a', a'') \in E^{j-1} \times E^{n-j}$  such that  $\text{int}_{\mathbb{C}} S(a', \cdot, a'') \neq \emptyset$  and define the  $n$ -fold generalized cross  $T := \mathbb{T}(E, \dots, E; E, \dots, E; S_1, \dots, S_n)$ , then there is a function  $f : T \setminus S \rightarrow \mathbb{C}$

which is separately holomorphic on  $T$  and there is no function  $\hat{f} \in \mathcal{M}(E^n)$  such that  $\hat{f} = f$  on  $T \setminus S$ .

**Problem.** Are the answers to Questions 1 and 3 positive if the condition on  $S$  is sharpened in the following form:  $S$  does not separate lower dimensional domains?

Finally, we state a result in positive direction.

**Theorem D.** For all  $j \in \{1, \dots, N\}$  ( $N \geq 2$ ), let  $D_j$  be a pseudoconvex domain in  $\mathbb{C}^{n_j}$  and let  $S$  be a relatively closed set of  $D := D_1 \times \dots \times D_N$  with  $\text{int } S = \emptyset$ . For  $j \in \{1, \dots, N\}$ , let  $S_j$  denote the set of all  $(a', a'') \in (D_1 \times \dots \times D_{j-1}) \times (D_{j+1} \times \dots \times D_N)$  such that  $\text{int}_{\mathbb{C}^{n_j}} S(a', \cdot, a'') \neq \emptyset$  and define the  $N$ -fold generalized cross  $T := \mathbb{T}(D_1, \dots, D_N; D_1, \dots, D_N; S_1, \dots, S_N)$ . Let  $f \in \mathcal{O}_s(T \setminus S)$  (resp.  $f \in \mathcal{M}_s(T \setminus S)$ ).

- (i) Then there are an open dense set  $\Omega$  of  $D$  and exactly one function  $\hat{f} \in \mathcal{O}(\Omega)$  such that  $\hat{f} = f$  on  $(T \cap \Omega) \setminus S$ .
- (ii) In the case where  $N = 2$ , (i) can be strengthened as follows. Let  $\Omega_j$  be a relatively compact pseudoconvex subdomain of  $D_j$  ( $j=1,2$ ). Then there are an open dense set  $A_j$  in  $\Omega_j$  and exactly one function  $\hat{f} \in \mathcal{O}(\hat{X})$  (resp.  $\hat{f} \in \mathcal{M}(\hat{X})$ ), where  $X := \mathbb{X}(A_1, A_2; \Omega_1, \Omega_2)$ , such that  $\hat{f} = f$  on  $(T \cap \hat{X}) \setminus S$ .

A remark is in order. In contrast with the other usual extension theorems (see [1], [2], [3], [4] and the references therein), the domain of meromorphic/holomorphic extension of the function  $f$  in Theorem D depends on  $f$ .

### 3. THREE COUNTEREXAMPLES

In the sequel we will fix a function  $v \in \mathcal{SH}(2E)$  such that  $v(0) = 0$  and the complete polar set  $\{z \in 2E : v = -\infty\}$  is dense in  $2E$ . For example one can choose  $v$  of the form

$$(3.1) \quad v(z) := \sum_{k=1}^{\infty} \frac{\log\left(\frac{|z-q_k|}{4}\right)}{d_k} - \sum_{k=1}^{\infty} \frac{\log\left(\frac{|q_k|}{4}\right)}{d_k},$$

where  $(\mathbb{Q} + i\mathbb{Q}) \cap 2E = \{q_1, q_2, \dots, q_k, \dots\}$ , and  $\{d_k\}_{k=1}^{\infty}$  is any sequence of positive real numbers such that  $\sum_{k=1}^{\infty} \frac{\log\left(\frac{|q_k|}{4}\right)}{d_k}$  is finite.

For any positive integer  $n \geq 2$ , define a new function  $u \in \mathcal{PSH}((2E)^n)$  and a subset  $\mathcal{A}$  of  $E^n$  as follows

$$(3.2) \quad u(z) := \sum_{k=1}^n v(z_k), \quad z = (z_1, \dots, z_n) \in (2E)^n,$$

$$\mathcal{A} = \mathcal{A}_n := \{z \in E^n : u(z) < -1\}.$$

Observe that  $\mathcal{A}$  is an open dense set of  $E^n$  because  $\mathcal{A}$  contains the set  $\{z \in E : v = -\infty\} \times \dots \times \{z \in E : v = -\infty\}$  which is dense in  $E^n$  by our construction (3.1) above.

**Proposition 3.1.** Let  $\mathcal{S}$  be any closed set contained in the closed set  $E^n \setminus \mathcal{A}$ . Then  $\mathcal{S}$  does not separate domains.

Taking this proposition for granted, we are now able to complete the proof of Proposition A.

*Proof of Proposition A.*

It is clear from (3.1) and (3.2) that the open dense set  $\mathcal{A}$  is plurithin at  $0 \in E^n$ . By Proposition 3.1, the closed set  $E^n \setminus \mathcal{A}$  does not separate domains. Therefore this set cannot contain any open set of a real hypersurface. Thus  $\mathcal{A}$  has all the desired properties.  $\square$

We now come back to Proposition 3.1.

*Proof.* One first observe that

$$\mathcal{S} \subset E^n \setminus \mathcal{A} = \{z \in E^n : u(z) \geq -1\}.$$

For any tuple of four vectors in  $\mathbb{R}^n$   $a := (a_1, \dots, a_n)$ ,  $b := (b_1, \dots, b_n)$ ,  $c := (c_1, \dots, c_n)$ ,  $d := (d_1, \dots, d_n)$  with the property that  $a_k < b_k$  and  $c_k < d_k$  for all  $k = 1, \dots, n$ , one defines the open cube of  $\mathbb{C}^n$

$$\Delta = \Delta(a, b, c, d) := \{z \in \mathbb{C}^n : a_k < \operatorname{Re} z_k < b_k, c_k < \operatorname{Im} z_k < d_k, k = 1, \dots, n\}.$$

It is clear that the intersection of two such cubes is either empty or a cube.

One first shows that for any cube  $\Delta \subset E^n$  the open set  $\Delta \setminus \mathcal{S}$  is connected. Indeed, pick two points  $z = (z_1, \dots, z_n)$  and  $w = (w_1, \dots, w_n)$  in  $\Delta \setminus \mathcal{S}$ . Since  $\{z \in E : v(z) = -\infty\}$  is dense in  $E$ , we can choose  $z' = (z'_1, \dots, z'_n)$  and  $w' = (w'_1, \dots, w'_n)$  in  $\Delta \setminus \mathcal{S}$  such that

- (i) the segments  $\gamma_1(t) := (1-t)z + tz'$  and  $\gamma_3(t) := (1-t)w + tw'$ ,  $0 \leq t \leq 1$ , are contained in  $\Delta \setminus \mathcal{S}$ ;
- (ii)  $z'_1, \dots, z'_n$  and  $w'_1, \dots, w'_n$  are in  $\{z \in E : v(z) = -\infty\}$ .

Consider now  $\gamma_2 : [0, 1] \longrightarrow \Delta$  given by

$$\gamma_2(t) := \left( w'_1, \dots, w'_j, (j+1-nt)z'_{j+1} + (nt-j)w'_{j+1}, z'_{j+2}, \dots, z'_n \right),$$

for  $t \in [\frac{j}{n}, \frac{j+1}{n}]$  and  $j = 0, \dots, n-1$ . By (3.2) and property (ii) above,  $\gamma_2(t) \in \{z \in E^n : u(z) = -\infty\}$  for all  $t \in [0, 1]$ . This implies that  $\gamma_2 : [0, 1] \longrightarrow \Delta \setminus \mathcal{S}$ .

Observe that  $\gamma_2(0) = z'$  and  $\gamma_2(1) = w'$ . By virtue of (i), the new path  $\gamma : [0, 1] \longrightarrow \Delta \setminus \mathcal{S}$  given by

$$\gamma(t) := \begin{cases} \gamma_1(3t), & t \in [0, \frac{1}{3}], \\ \gamma_2(3t-1), & t \in [\frac{1}{3}, \frac{2}{3}], \\ \gamma_3(3t-2), & t \in [\frac{2}{3}, 1]. \end{cases}$$

satisfies  $\gamma(0) = z$  and  $\gamma(1) = w$ , and  $\Delta \setminus \mathcal{S}$  is therefore connected.

Now let  $U$  be any subdomain of  $E^n$ . We wish to show that  $U \setminus \mathcal{S}$  is connected. To do this, pick points  $z = (z_1, \dots, z_n)$  and  $w = (w_1, \dots, w_n)$  in  $U \setminus \mathcal{S}$ . Since  $U$  is arcwise connected, there is a continuous function  $\gamma : [0, 1] \longrightarrow U$  such that  $\gamma(0) = z$  and  $\gamma(1) = w$ .

By the Heine-Borel Theorem, the compact set  $\mathcal{L} := \gamma([0, 1])$  can be covered by a finite number of cubes  $\Delta_l$  ( $1 \leq l \leq N$ ) with  $\Delta_l \subset U$  and  $\Delta_l \cap \mathcal{L} \neq \emptyset$ . Since the path  $\mathcal{L}$  is connected, the union  $\bigcup_{l=1}^N \Delta_l$  is also connected.

Suppose without loss of generality that  $z \in \Delta_1$  and  $w \in \Delta_N$ . From the discussion above, if  $\Delta_1 \cap \Delta_2 \neq \emptyset$  then  $(\Delta_1 \setminus \mathcal{S}) \cap (\Delta_2 \setminus \mathcal{S}) = (\Delta_1 \cap \Delta_2) \setminus \mathcal{S}$  is connected, and hence  $(\Delta_1 \cup \Delta_2) \setminus \mathcal{S}$  is also connected. Repeating this argument at most  $N$  times and using the connectivity of  $\bigcup_{l=1}^N \Delta_l$ , we finally conclude that  $\bigcup_{l=1}^N \Delta_l \setminus \mathcal{S} (\subset U \setminus \mathcal{S})$  is also connected. This completes the proof.  $\square$

**Corollary 3.2.** (i) *If  $\mathcal{S}_1, \dots, \mathcal{S}_N$  are relatively closed subsets of  $E^n$  which do not separate domains, then the union  $\bigcup_{l=1}^N \mathcal{S}_l$  does not separate domains too.*  
(ii) *Let  $\mathcal{A}, \mathcal{S}$  be as in Proposition 3.1. Then for any closed sets  $F_1$  in  $\mathbb{C}^p$  and  $F_2$  in  $\mathbb{C}^q$  ( $p, q \geq 0$ ), the closed set  $F_1 \times \mathcal{S} \times F_2$  does not separate domains in  $\mathbb{C}^p \times E^n \times \mathbb{C}^q$ .*

*Proof.* To prove part (i), let  $U$  be any subdomain of  $E^n$ . Since  $U \setminus \left(\bigcup_{l=1}^N \mathcal{S}_l\right) = ((U \setminus \mathcal{S}_1) \cdots \setminus \mathcal{S}_N)$ , part (i) follows from the hypothesis of  $\mathcal{S}_l$ .

To prove part (ii), consider any subdomain  $U$  of  $\mathbb{C}^p \times E^n \times \mathbb{C}^q$  and let  $(z_1, w_1, t_1), (z_2, w_2, t_2)$  be two points in  $U \setminus (F_1 \times \mathcal{S} \times F_2)$ . Since  $\mathcal{A}$  is an open dense set of  $E^n$ ,  $\text{int}(F_1 \times \mathcal{S} \times F_2) = \emptyset$ , and therefore we are able to perform the compact argument that we had already used in the proof of Proposition 3.1. Consequently, one is reduced to the case where  $U$  is a cube of  $\mathbb{C}^{p+n+q}$ .

Another reduction is in order. Since  $U \setminus (F_1 \times \mathcal{S} \times F_2)$  is open and  $\mathcal{A}$  is dense in  $E^n$ , by replacing  $w_1$  (resp.  $w_2$ ) by  $w'_1$  (resp.  $w'_2$ ) close to  $w_1$  (resp.  $w_2$ ), we may suppose that  $w_1, w_2 \in E^n \setminus \mathcal{S}$ .

Write the cube  $U$  as the product of  $\Delta_1 \times \Delta_2 \times \Delta_3$ , where  $\Delta_1$  (resp.  $\Delta_2$  and  $\Delta_3$ ) is a cube in  $\mathbb{C}^p$  (resp.  $\mathbb{C}^n$  and  $\mathbb{C}^q$ ). By Proposition 3.1, there is a continuous path  $\gamma_2 : [0, 1] \longrightarrow \Delta_2 \setminus \mathcal{S}$  such that  $\gamma_2(0) = w_1$  and  $\gamma_2(1) = w_2$ .

We now consider the path  $\gamma : [0, 1] \longrightarrow \Delta_1 \times \Delta_2 \times \Delta_3 \setminus \mathcal{S}$ , where  $\gamma(t) := (\gamma_1(t), \gamma_2(t), \gamma_3(t))$  and  $\gamma_1(t) := (1-t)z_1 + tz_2$ ,  $\gamma_3(t) := (1-t)t_1 + tt_2$ ,  $t \in [0, 1]$ . It is easy to see that  $\gamma(0) = (z_1, w_1, t_1)$  and  $\gamma(1) = (z_2, w_2, t_2)$ , which finishes the proof.  $\square$

The following two lemmas will be crucial for the proof of Propositions B and C.

**Lemma 3.3.** *For an open set  $\Omega \subset \mathbb{C}^n$  and  $A \subset \Omega$ , we have either  $\omega_{A,\Omega} \equiv 0$  or  $\sup_{\Omega} \omega_{A,\Omega} = 1$ .*

*Proof.* We first prove the lemma in the case where  $\Omega$  is bounded. Suppose in order to get a contradiction that  $\sup_{\Omega} h_{A,\Omega}^* = M$  with  $0 < M < 1$ . By virtue of the definition of  $h_{A,\Omega}^*$ , it follows that

$$\begin{aligned} \{u : u \in \mathcal{PSH}(\Omega), u \leq 1 \text{ on } \Omega, u \leq 0 \text{ on } A\} \\ = \{u : u \in \mathcal{PSH}(\Omega), u \leq M \text{ on } \Omega, u \leq 0 \text{ on } A\}. \end{aligned}$$

Therefore,  $h_{A,\Omega}^*(z) < M h_{A,\Omega}^* < h_{A,\Omega}^*$  for any  $z \in \Omega$  with  $h_{A,\Omega}^*(z) > 0$ , and we obtain the desired contradiction.

The general case is analogous using the definition of  $\omega_{A,\Omega}$  and the Hartog's Lemma.  $\square$

**Lemma 3.4.** *Let  $\Omega_1 \subsetneq \Omega_2$  be two domains of  $\mathbb{C}^n$  such that  $\Omega_2$  is pseudoconvex. Assume that there is a upper bounded function  $\phi \in \mathcal{PSH}(\Omega_2)$  satisfying*

$\Omega_1 = \{z \in \Omega_2 : \phi(z) < 0\}$ . Then there is a function  $f \in \mathcal{O}(\Omega_1)$  such that there is no function  $\hat{f} \in \mathcal{M}(\Omega_2)$  verifying  $\hat{f} = f$  on  $\Omega_1$ .

*Proof.* It is clear from the hypothesis that  $\Omega_1$  is also pseudoconvex. Let  $\partial\Omega_1$  be the boundary of  $\Omega_1$  in  $\Omega_2$  and let  $S$  be a countable dense subset of  $\partial\Omega_1$ . It is a classical fact that there is a function  $f \in \mathcal{O}(\Omega_1)$  such that

$$(3.3) \quad \lim_{z \in \Omega_1, z \rightarrow w} |f(z)| = \infty, \quad w \in S.$$

We will show that this is the desired function. Indeed, suppose in order to get a contradiction that there is a function  $\hat{f} \in \mathcal{M}(\Omega_2)$  verifying  $\hat{f} = f$  on  $\Omega_1$ . Because of (3.3),  $S$  and then  $\partial\Omega_1$  are contained in the pole set of  $\hat{f}$  (i.e. the union of the set of all poles of  $\hat{f}$  and the set of all indeterminacy points of  $\hat{f}$ ). Therefore, for any point  $w \in \partial\Omega_1$ , there is a small open neighborhood  $U$  of  $w$  and a complex analytic subset of codimension one  $C$  such that  $U \setminus C \subset \Omega_1$ . Since  $\phi \in \mathcal{PSH}(U)$  is upper bounded,  $\phi(w) = \limsup_{z \in U \setminus C, z \rightarrow w} \phi(z) = \phi(w) \leq 0$  for all  $w \in \partial\Omega_1$ . Since  $\Omega_1 \subsetneq \Omega_2$ ,  $\phi$  is non-constant and therefore  $\phi(w) < 0$  for all  $w \in \partial\Omega_1$ , which is a contradiction.  $\square$

We are now ready to prove Propositions B and C.

*The proof of Proposition B.* Suppose, without loss of generality, that  $D = E^p$  and  $G = E^q$ . The general case is almost analogous. Let  $F_p$  (resp.  $F_q$ ) be any closed ball contained in the open set  $\mathcal{A}_p$  (resp.  $\mathcal{A}_q$ ). We now define the relatively closed set  $S$  by the formula

$$(3.4) \quad S := (E^p \setminus \mathcal{A}_p) \times F_q \bigcup F_p \times (E^q \setminus \mathcal{A}_q).$$

We now check the properties (i) and (ii) of Proposition B. First,  $\text{int } S = \emptyset$  because  $\mathcal{A}_p$  (resp.  $\mathcal{A}_q$ ) is open dense set in  $E^p$  (resp.  $E^q$ ). Second, by Proposition 3.1 and Corollary 3.2(ii), the two relatively closed sets  $(E^p \setminus \mathcal{A}_p) \times F_q$  and  $F_p \times (E^q \setminus \mathcal{A}_q)$  do not separate domains. By Corollary 3.2(i), the union  $S$  also enjoys this property. Thus  $S$  satisfies (i).

Using (3.4), a direct computation gives that  $A = \mathcal{A}_p$  and  $B = \mathcal{A}_q$  and  $A, B$  are open, in particular they are locally pluriregular.

By the classical cross theorem (see for instance [7] or [1]), the envelope of holomorphy of  $X$  is given by

$$\widehat{X} := \{(z, w) \in E^p \times E^q : \omega_{A, E^p}(z) + \omega_{B, E^q}(w) < 1\}.$$

We now show that  $h_{\mathcal{A}_n, E^n}^*(0) > 0$  for  $n \geq 2$ . Indeed, let  $M := \sup_{E^n} u$ , where  $u$  is defined in (3.2). Observe that  $M > 0$  since  $u(0) = 0$ . Consider the function  $\tilde{u} \in \mathcal{PSH}(E^n)$  given by

$$\tilde{u}(z) := \frac{u(z) - M}{M + \frac{1}{2}} + 1, \quad \text{for } z \in E^n.$$

It can be easily checked that  $\tilde{u}(z) \leq 1$  on  $E^n$  and  $\tilde{u}(z) \leq 0$  on  $\mathcal{A}_n$ . Thus  $\tilde{u}(0) \leq h_{\mathcal{A}_n, E^n}^*(0)$ . On the other hand,  $\tilde{u}(0) = \frac{1}{2M+1} > 0$ . Hence our assertion above follows.

We next show that  $\widehat{X} \subsetneq E^p \times E^q$ . Indeed, we have

$$\left\{w \in E^q : (0, w) \in \widehat{X}\right\} \subset \left\{w \in E^q : h_{\mathcal{A}_q, E^q}^*(w) < 1 - h_{\mathcal{A}_p, E^p}^*(0)\right\}.$$

Since  $h_{\mathcal{A}_q, E^q}^*(w) > 0$  and  $h_{\mathcal{A}_p, E^p}^*(0) > 0$ , Lemma 3.3 applies and consequently the latter set is strictly contained in  $E^q$ . This proves our assertion above.

We are now ready to complete the proof. By Lemma 3.4, there is a holomorphic function  $f$  in  $\widehat{X}$  which cannot be meromorphically extended to  $E^p \times E^q$ . Therefore, there is no meromorphic function  $\hat{f} \in \mathcal{M}(E^p \times E^q)$  such that  $\hat{f} = f$  on the set of unicity for meromorphic functions

$$X \setminus S = ((\mathcal{A}_p \setminus F_p) \times E^q) \cup (F_p \times \mathcal{A}_q) \cup (E^p \times (\mathcal{A}_q \setminus F_q)) \cup (\mathcal{A}_p \times F_q).$$

The proof is thereby finished.  $\square$

*The proof of Proposition C.* In order to simplify the notation, we only consider the case  $n = 3$ , the general case  $n > 3$  is analogous. Let  $\mathcal{B}$  be the following open dense subset of  $E$

$$\mathcal{B} := \left\{z \in E : v(z) < -\frac{1}{2}\right\},$$

where  $v$  is given by (3.1). Then by virtue of (3.2), it can be checked that  $(E \setminus \mathcal{B}) \times (E \setminus \mathcal{B}) \subset E^2 \setminus \mathcal{A}_2$ . Fix any closed ball  $F$  contained in the open set  $\mathcal{B}$ . Next one applies Proposition 3.1 and Corollary 3.2 to the relatively closed set  $\mathcal{S} := (E \setminus \mathcal{B}) \times (E \setminus \mathcal{B})$ . Consequently, the set

(3.5)

$$S := (F \times (E \setminus \mathcal{B}) \times (E \setminus \mathcal{B})) \cup ((E \setminus \mathcal{B}) \times F \times (E \setminus \mathcal{B})) \cup ((E \setminus \mathcal{B}) \times (E \setminus \mathcal{B}) \times F)$$

does not separate domains in  $E^3$ . Moreover, since  $\mathcal{B}$  is an open dense subset of  $E$ , we see that  $\text{int } S = \emptyset$  and  $S$  is relatively closed. Hence  $S$  satisfies property (i).

To verify (ii), one first computes the following sets using (3.5)

(3.6)

$$S_1 = S_2 = S_3 = (E \setminus \mathcal{B}) \times (E \setminus \mathcal{B}), \quad T = (\mathcal{B} \times E \times E) \cup (E \times \mathcal{B} \times E) \cup (E \times E \times \mathcal{B}).$$

Next, by the product property for the relative extremal function [6], we have  $h_{\mathcal{B} \times \mathcal{B}, E^2}^*(0) = h_{\mathcal{B}, E}^*(0)$ . Since  $\mathcal{B} \times \mathcal{B} \subset \mathcal{A}_2$  and we have shown in Proposition B that  $h_{\mathcal{A}_2, E^2}^*(0) > 0$ , it follows that  $h_{\mathcal{B}, E}^*(0) > 0$ .

Consider now the domain of holomorphy

$$(3.7) \quad \Omega := \{(z, w, t) \in E^3 : h_{\mathcal{B}, E}^*(z) + h_{\mathcal{B}, E}^*(w) + h_{\mathcal{B}, E}^*(t) < 2\}.$$

Since  $\mathcal{B}$  is open and therefore locally pluriregular, it can be proved using Lemma 5 in [2] that  $\Omega$  is a domain. Moreover it can be easily checked that  $T \subset \Omega$  using (3.6) and (3.7).

We now prove that  $\Omega \subsetneq E^3$ . Indeed, since  $h_{\mathcal{B}, E}^*(0) > 0$ , by Lemma 3.3 there are  $z, w \in E$  such that  $h_{\mathcal{B}, E}^*(z) > \frac{2}{3}$ ,  $h_{\mathcal{B}, E}^*(w) > \frac{2}{3}$ . Then the fiber

$$\{t \in E : (z, w, t) \in \Omega\} \subset \left\{t \in E : h_{\mathcal{B}, E}^*(t) < \frac{2}{3}\right\}.$$

Another application of Lemma 3.3 shows that the latter set is strictly contained in  $E$ . This proves our assertion from above.



We are now ready to complete the proof. By Lemma 3.4, there is a holomorphic function  $f$  in  $\Omega$  which cannot be meromorphically extended to  $E^n$ . Therefore, there is no meromorphic function  $\hat{f} \in \mathcal{M}(E^n)$  such that  $\hat{f} = f$  on the set of unicity for meromorphic functions  $T \setminus S$ . Hence, the proof is finished.  $\square$

#### 4. AUXILIARY RESULTS

Let  $S$  be a subset of an open set  $D \subset \mathbb{C}^n$ . Then  $S$  is said to be of *Baire category I* if  $S$  is contained in a countable union of relatively closed sets in  $D$  with empty interior. Otherwise,  $S$  is said to be of *Baire category II*.

The following lemma is very useful.

**Lemma 4.1.** *For  $j \in \{1, \dots, M\}$  and  $M \geq 2$ , let  $\Omega_j$  be a domain in  $\mathbb{C}^{m_j}$  and let  $S$  be a relatively closed set of  $\Omega_1 \times \dots \times \Omega_M$  with  $\text{int } S = \emptyset$ . For  $a_j \in \Omega_j$ ,  $j \in \{3, \dots, M\}$ , let  $S(a_3, \dots, a_M)$  denote the set of all  $a_2 \in \Omega_2$  such that  $\text{int}_{\mathbb{C}^{m_1}} S(\cdot, a_2, a_3, \dots, a_M) = \emptyset$ . For  $j \in \{4, \dots, M\}$ , let  $S(a_j, \dots, a_M)$  denote the set of all  $a_{j-1} \in \Omega_{j-1}$  such that  $\Omega_{j-2} \setminus S(a_{j-1}, a_j, \dots, a_M)$  is of Baire category I, and finally let  $\mathcal{S}$  denote the set of all  $a_M \in \Omega_M$  such that  $\Omega_{M-1} \setminus S(a_M)$  is of Baire category I. Then  $\Omega_M \setminus \mathcal{S}$  is of Baire category I.*

*Proof.* For  $j \in \{1, \dots, M\}$  let  $(\mathbb{Q} + i\mathbb{Q})^{m_j} = \{q_1^j, \dots, q_n^j, \dots\}$  and  $\delta_n := \frac{1}{n}$ ,  $n \in \mathbb{N}$ . For  $q \in \Omega_j$  and  $r > 0$ , let  $\Delta_q(r)$  denote the polydisc in  $\mathbb{C}^{m_j}$  with center  $q$  and multi-radius  $(r, \dots, r)$ .

Suppose in order to get a contradiction that  $\Omega_M \setminus \mathcal{S}$  is of Baire category II. Then for all  $a_M \in \Omega_M \setminus \mathcal{S}$ ,  $\Omega_{M-1} \setminus S(a_M)$  is of Baire category II. Therefore, for  $j = M-1, \dots, 3$  and any  $a_j \in \Omega_j \setminus S(a_{j+1}, \dots, a_M)$ , the set  $\Omega_{j-1} \setminus S(a_j, \dots, a_M)$  is of Baire category II. Put

$$S_n(a_3, \dots, a_M) := \{a_2 \in \Omega_2 : S(\cdot, a_2, a_3, \dots, a_M) \supset \Delta_{q_n^1}(\delta_n)\}.$$

Since  $S$  is relatively closed,  $S_n(a_3, \dots, a_M)$  is also relatively closed in  $\Omega_2$ . Moreover, from the definition of  $S(a_3, \dots, a_M)$ , we have the following identity

$$\Omega_2 \setminus S(a_3, \dots, a_M) = \bigcup_{n=1}^{\infty} S_n(a_3, \dots, a_M).$$

Since it is shown in the above discussion, that  $\Omega_2 \setminus S(a_3, \dots, a_M)$  is of Baire category II in  $\Omega_2$ , we can therefore apply the Baire Theorem to the right side of the latter identity. Consequently, there exist  $n_1, n_2 \in \mathbb{N}$  such that  $S_{n_1}(a_3, \dots, a_M) \supset \Delta_{q_{n_2}^2}(\delta_{n_2})$ . This implies that  $S(\cdot, \cdot, a_3, \dots, a_M) \supset \Delta_{q_{n_1}^1}(\delta_{n_1}) \times \Delta_{q_{n_2}^2}(\delta_{n_2})$ .

Now, define inductively for  $j = 2, \dots, M-1$  and  $n_1, \dots, n_j \in \mathbb{N}$ ,

$$\begin{aligned} & S_{n_1, \dots, n_j}(a_{j+2}, \dots, a_M) \\ &:= \left\{ a_{j+1} \in \Omega_{j+1} : S(\dots, a_{j+1}, \dots, a_M) \supset \Delta_{q_{n_1}^1}(\delta_{n_1}) \times \dots \times \Delta_{q_{n_j}^j}(\delta_{n_j}) \right\}. \end{aligned}$$

Since  $S$  is relatively closed,  $S_{n_1, \dots, n_j}(a_{j+2}, \dots, a_M)$  is also relatively closed. Moreover, it can be checked that

$$\Omega_{j+1} \setminus S(a_{j+2}, \dots, a_M) \subset \bigcup_{n_1, \dots, n_j=1}^{\infty} S_{n_1, \dots, n_j}(a_{j+2}, \dots, a_M).$$

Applying the Baire Theorem again, it follows that there are  $n_1, \dots, n_{j+1} \in \mathbb{N}$  such that  $S_{n_1, \dots, n_j}(a_{j+2}, \dots, a_M) \supset \Delta_{q_{n_{j+1}}^{j+1}}(\delta_{n_{j+1}})$ , and hence

$$S(\dots, a_{j+2}, \dots, a_M) \supset \Delta_{q_{n_1}^1}(\delta_{n_1}) \times \dots \times \Delta_{q_{n_{j+1}}^{j+1}}(\delta_{n_{j+1}}).$$

Finally, we obtain for  $j = M - 1$  that  $\text{int } S \neq \emptyset$ , which contradicts the hypothesis. Hence, the proof is complete.  $\square$

**Remark 4.2.** *If we apply Lemma 4.1 to the case where  $\Omega_1 := D_j$  and  $\Omega_2 := (D_1 \times \dots \times D_{j-1}) \times (D_{j+1} \times \dots \times D_N)$ . Then, for each  $j \in \{1, \dots, N\}$ , the set  $S_j$  in the statement of Theorem D is of Baire category I. In particular, the set  $\Omega \setminus ((T \setminus S) \cap \Omega)$  is of Baire category I for all open sets  $\Omega \subset D$ .*

**Lemma 4.3.** *Let  $U \subset \mathbb{C}^p$  and  $V \subset \mathbb{C}^q$  be two pseudoconvex domains. Consider four sets  $C \subset A \subset U$  and  $D \subset B \subset V$  such that  $\overline{C} = \overline{A}$ ,  $\overline{D} = \overline{B}$  and  $\overline{A}$ ,  $\overline{B}$  are locally pluriregular. Put  $X := \mathbb{X}(A, B; U, V)$  and  $\hat{X} := \widehat{\mathbb{X}}(\overline{A}, \overline{B}; U, V)$ . Assume  $f \in \mathcal{O}_s(X)$  and there is a finite constant  $K$  such that for all  $c \in C$  and  $d \in D$ ,*

$$\sup_V |f(c, \cdot)| < K \quad \text{and} \quad \sup_U |f(\cdot, d)| < K.$$

*Then there exists a unique function  $\hat{f} \in \mathcal{O}(\hat{X})$  such that  $\hat{f} = f$  on  $\hat{X} \cap X$ .*

*Proof.* From the hypothesis on the boundedness of  $f$ , it follows that the two families  $\{f(c, \cdot) : c \in C\}$  and  $\{f(\cdot, d) : d \in D\}$  are normal. We now define two functions  $f_1$  on  $\overline{A} \times V$  and  $f_2$  on  $U \times \overline{B}$  as follows.

For any  $z \in \overline{A}$ , choose a sequence  $(c_n)_{n=1}^{\infty} \subset C$  such that  $\lim_{n \rightarrow \infty} c_n = z$  and the sequence  $(f(c_n, \cdot))_{n=1}^{\infty}$  converges uniformly on compact subsets of  $V$ . We let

$$f_1(z, w) := \lim_{n \rightarrow \infty} f(c_n, w), \quad \text{for all } w \in V.$$

Similarly, for any  $w \in \overline{B}$ , choose a sequence  $(d_n)_{n=1}^{\infty} \subset D$  such that  $\lim_{n \rightarrow \infty} d_n = w$  and the sequence  $(f(\cdot, d_n))_{n=1}^{\infty}$  converges uniformly on compact subsets of  $U$ . We let

$$f_2(z, w) := \lim_{n \rightarrow \infty} f(z, d_n), \quad \text{for all } z \in U.$$

We first check that  $f_1$  and  $f_2$  are well-defined. Indeed, it suffices to verify this for  $f_1$  since the same argument also applies to  $f_2$ . Let  $(c'_n)_{n=1}^{\infty} \subset C$  be another sequence such that  $\lim_{n \rightarrow \infty} c'_n = z$  and the sequence  $(f(c'_n, \cdot))_{n=1}^{\infty}$  converges uniformly on compact subsets of  $V$ . Since for all  $b \in B$ ,

$$\lim_{n \rightarrow \infty} f(c_n, b) = f(z, b) = \lim_{n \rightarrow \infty} f(c'_n, b),$$

and since  $B$  is the set of unicity for holomorphic functions on  $V$ , our claim follows.

One next verifies that  $f_1 = f_2$  on  $\overline{A} \times \overline{B}$ . Indeed, let  $z \in \overline{A}$ ,  $w \in \overline{B}$  and let  $(c_n)_{n=1}^\infty \subset C$ ,  $(d_n)_{n=1}^\infty \subset D$  be as above. Then clearly, we have

$$f_1(z, w) = \lim_{n \rightarrow \infty} f(c_n, d_n) = f_2(z, w).$$

We are now able to define a function  $\tilde{f}$  on  $\mathbb{X}(\overline{A}, \overline{B}; U, V)$  by the formula  $f = f_1$  on  $\overline{A} \times V$  and  $f = f_2$  on  $U \times \overline{B}$ . It follows from the construction of  $f_1$  and  $f_2$  that  $\tilde{f} \in \mathcal{O}_s(\mathbb{X}(\overline{A}, \overline{B}; U, V))$ .

One next checks that  $\tilde{f} = f$  on  $X$ . Indeed, since for each  $a \in A$ ,  $f(a, \cdot)$  and  $\tilde{f}(a, \cdot)$  are holomorphic, it suffices to verify that  $\tilde{f}(a, d) = f(a, d)$ . But the latter equality follows easily from the definition of  $f_1$  and the hypothesis.

Finally, one applies the classical cross theorem (cf. [7], [1]) to  $\tilde{f} \in \mathcal{O}_s(\mathbb{X}(\overline{A}, \overline{B}; U, V))$ , thus the existence of  $\hat{f}$  follows. The unicity of  $\hat{f}$  is also clear.  $\square$

**Lemma 4.4.** (Rothstein type theorem, cf. [8]). *Let  $f \in \mathcal{O}(E^p \times E^q)$ . Assume that  $A \subset E^p$  such that for all open subsets  $U \subset E^p$ ,  $A \cap U$  is of Baire category II and for all  $z \in E^p$  we have  $(P_f)(z, \cdot) \neq E^q$ . Here  $P_f$  denotes the pole set of  $f$ . Let  $G \subset \mathbb{C}^q$  be a domain such that  $E^q \subset G$  and assume that for all  $a \in A$ , the function  $f(a, \cdot)$  extends meromorphically to  $\widetilde{f(a, \cdot)} \in \mathcal{M}(G)$ . Then for any relatively compact subdomain  $\tilde{G} \subset G$ , there are an open dense set  $\mathcal{A} \subset E^p$  and a function  $\tilde{f} \in \mathcal{M}(\Omega)$ , where  $\Omega := E^p \times E^q \cup \mathcal{A} \times \tilde{G}$  such that  $\tilde{f} = f$  on  $E^p \times E^q$ .*

*Proof.* We present a sketch of the proof.

(1) The case where  $G := \Delta_0(R)$  ( $R > 1$ ).

Arguing as in the proof of Rothstein's theorem given in [10], the conclusion of the lemma follows.

(2) The general case, where  $G$  is arbitrary.

Fix an  $a \in E^p$  and  $r > 0$ . Let  $B$  denote the set of all  $b \in G$  such that there exist  $0 < r_b < r$ , an open dense  $A_b$  of  $\Delta_a(r_b)$  and  $f_b \in \mathcal{M}(A_b \times \Delta_b(r_b))$  such that for all  $\alpha \in A \cap A_b$ ,  $f_b(\alpha, \cdot) = \widetilde{f(\alpha, \cdot)}$  on  $\Delta_b(r_b)$ .

Obviously,  $B$  is open. Using the case (1) and the hypothesis on  $A$ , one can show that  $B$  is closed in  $G$ . Thus  $B = G$ . Moreover, one can also show that if  $A_b \cap A_{b'} \neq \emptyset$  and  $\Delta_b(r_b) \cap \Delta_{b'}(r_{b'}) \neq \emptyset$ , then  $f_b = f_{b'}$  on  $(A_b \cap A_{b'}) \times (\Delta_b(r_b) \cap \Delta_{b'}(r_{b'}))$ . Therefore, using the hypothesis that  $\tilde{G}$  is relatively compact, we see that for any  $a \in E^p$  and any  $r > 0$ , there is an open set  $\mathcal{A}_{a,r} \subset \Delta_a(r)$  and  $f_{a,r} \in \mathcal{M}(\mathcal{A}_{a,r} \times \tilde{G})$

such that for all  $\alpha \in A \cap \mathcal{A}_{a,r}$ ,  $f_{a,r}(\alpha, \cdot) = \widetilde{f(\alpha, \cdot)}$  on  $\tilde{G}$ .

Finally, let  $\mathcal{A} := \bigcup_{a \in E^p, r > 0} \mathcal{A}_{a,r}$ . This open set is clearly dense in  $E^p$ . By gluing the function  $f_{a,r}$  together, we obtain the desired meromorphic extension  $\tilde{f} \in \mathcal{M}(\Omega)$ ; so the proof of the lemma is completed.  $\square$

## 5. PROOF OF THEOREM D

We will only give the proof of Theorem D for the case where  $f$  is separately meromorphic. Since the case where  $f$  is separately holomorphic is quite similar and in some sense simpler, it is therefore left to the reader.

**Proof of Part (ii).**

Put

$$(5.1) \quad \mathcal{A}_j := \{z_j \in D_j : \text{int}_{\mathbb{C}^{n_j}} S(z_j, \cdot) = \emptyset\} \quad \text{for } j = 1, 2.$$

By Lemma 4.1,  $D_j \setminus \mathcal{A}_j$  is of Baire category I. For  $a_j \in \mathcal{A}_j$  ( $j = 1, 2$ ), let  $\widetilde{f(a_1, \cdot)}$  (resp.  $\widetilde{f(\cdot, a_2)}$ ) denote the meromorphic extension of  $f(a_1, \cdot)$  (resp.  $f(\cdot, a_2)$ ) to  $D_2$  (resp. to  $D_1$ ).

Let  $\mathcal{U} \subset D_1$ ,  $\mathcal{V} \subset D_2$  be arbitrary open sets. For a relatively compact pseudoconvex subdomain  $V$  of  $\mathcal{V}$  and for a positive number  $K$ , let  $Q_{V,K}^1$  denote the set of  $a_1 \in \mathcal{A}_1 \cap \mathcal{U}$  such that  $\sup_V |\widetilde{f(a_1, \cdot)}| \leq K$  (and thus  $\widetilde{f(a_1, \cdot)} \in \mathcal{O}(V)$ ). By virtue of (5.1) and the hypothesis, a countable number of the  $Q_{V,K}^1$  cover  $\mathcal{A}_1 \cap \mathcal{U}$ . Since the latter set is of Baire category II, we can choose  $V, K_1$  such that the closure  $\overline{Q_{V,K_1}^1}$  contains a polydisc  $\Delta_1 \subset \mathcal{U}$  and  $Q_{V,K_1}^1 \cap \Delta_1$  is of Baire category II in  $\Delta_1$ .

For a relatively compact pseudoconvex subdomain  $U$  of  $\Delta_1$  and for a positive number  $K$ , we denote by  $Q_{U,K}^2$  the set of  $a_2 \in \mathcal{A}_2 \cap V$  such that  $\sup_U |\widetilde{f(\cdot, a_2)}| \leq K$  (and thus  $\widetilde{f(\cdot, a_2)} \in \mathcal{O}(U)$ ). By virtue of (5.1) and the hypothesis, a countable number of the  $Q_{U,K}^2$  cover  $\mathcal{A}_2 \cap V$ . Since the latter set is of Baire category II, we can choose  $U, K_2$  such that  $\overline{Q_{U,K_2}^2}$  contains a polydisc  $\Delta_2 \subset V$  and  $Q_{U,K_2}^2 \cap \Delta_2$  is of Baire category II in  $\Delta_2$ .

Now let  $K := \max\{K_1, K_2\}$ ,  $A := \mathcal{A}_1 \cap U$ ,  $C := Q_{V,K}^1 \cap U$ ,  $B := \mathcal{A}_2 \cap \Delta_2$ ,  $D := Q_{V,K}^2 \cap \Delta_2$ . Then it is easy to see that  $\overline{A} = \overline{C} = U$  and  $\overline{B} = \overline{D} = \Delta_2$ . Moreover, all other hypotheses of Lemma 4.3 are fulfilled. Consequently, an application of this lemma gives the following.

*Let  $\mathcal{U} \subset D_1$ ,  $\mathcal{V} \subset D_2$  be arbitrary open sets. Then there is a polydisc  $\Delta_a(r) \subset \mathcal{U} \times \mathcal{V}$  and a function  $\hat{f} \in \mathcal{O}(\Delta_a(r))$  such that  $\hat{f} = f$  on  $(T \setminus S) \cap \Delta_a(r)$ .*

Write  $a = (a_1, a_2) \in D_1 \times D_2$ . Since the set  $\mathcal{A}_j \cap \Delta_{a_j}(r)$  is of Baire category II and by replacing  $\Delta_{a_j}(r)$  by a smaller polydisc, we see that this set satisfies the hypothesis of Lemma 4.4. Consequently, an application of this lemma gives  $f_a^1 \in \mathcal{M}(\Delta_{a_1}(r) \times \Omega_2)$  and  $f_a^2 \in \mathcal{M}(\Omega_1 \times \Delta_{a_2}(r))$  which coincide with  $f$  on  $(T \setminus S) \cap \Delta_a(r)$ . Moreover, one sees that the function  $f_{\mathcal{U},\mathcal{V}}$  given by

$$f_{\mathcal{U},\mathcal{V}} := f_a^1 \text{ on } \Delta_{a_1}(r) \times \Omega_2, \text{ and } f_{\mathcal{U},\mathcal{V}} := f_a^2 \text{ on } \Omega_1 \times \Delta_{a_2}(r),$$

is well-defined, meromorphic on the cross  $X := \mathbb{X}(\Delta_{a_1}(r), \Delta_{a_2}(r); \Omega_1, \Omega_2)$ , and  $f_{\mathcal{U},\mathcal{V}} = f$  on  $(T \setminus S) \cap X$ .

Using Remark 4.2, one can also prove the following. If  $\mathcal{U}' \subset D_1$ ,  $\mathcal{V}' \subset D_2$  are arbitrary open sets and  $f_{\mathcal{U}',\mathcal{V}'}$  is the corresponding meromorphic function defined on the corresponding cross  $X'$ , then  $f_{\mathcal{U},\mathcal{V}} = f_{\mathcal{U}',\mathcal{V}'}$  on  $X \cap X'$ .

Let  $A_j := \bigcup_{\mathcal{U} \subset \Omega_1, \mathcal{V} \subset \Omega_2} \Delta_{a_j}(r)$ , for  $j = 1, 2$ . It is clear that  $A_j$  is an open dense set in  $\Omega_j$ . Then gluing all  $f_{\mathcal{U},\mathcal{V}}$ , we obtain a function  $\hat{f}$  meromorphic on  $X := \mathbb{X}(A_1, A_2; \Omega_1, \Omega_2)$  satisfying  $\hat{f} = f$  on  $(T \setminus S) \cap X$ . Finally, one applies Theorem 1.3 in [4] to  $\hat{f}$ , and the conclusion of Part (ii) follows.

**Proof of Part (i).**

In the sequel,  $\Sigma_M$  will denote the group of permutation of  $M$  elements  $\{1, \dots, M\}$ . Moreover, for any  $\sigma \in \Sigma_M$  and under the hypothesis and the notation of Lemma 4.1, we define

$$S^\sigma := \{z^\sigma : z \in S\} \quad \text{and} \quad \Omega^\sigma := \Omega_{\sigma(1)} \times \dots \times \Omega_{\sigma(M)},$$

where

$$z^\sigma := (z_{\sigma(1)}, \dots, z_{\sigma(M)}), \quad z \in \Omega = \Omega_1 \times \dots \times \Omega_M.$$

If in the statement of Lemma 4.1, one replaces  $S$  by  $S^\sigma$  and  $\Omega$  by  $\Omega^\sigma$ , then one obtains  $\mathcal{S}^\sigma$ ,  $S^\sigma(a_{\sigma(N)}), \dots, S^\sigma(a_{\sigma(3)}), \dots, a_{\sigma(N)}$ . The proof will be divided into three steps.

**Step 1:  $N = 2$ .**

By virtue of Part (ii), for each pair of relatively compact pseudoconvex subdomains  $\Omega_j \subset D_j$  ( $j = 1, 2$ ) we obtain a polydisc  $\Delta_{\Omega_1, \Omega_2} \subset \Omega_1 \times \Omega_2$  and a function  $f_{\Omega_1, \Omega_2} \in \mathcal{O}(\Delta_{\Omega_1, \Omega_2})$  such that  $f = f_{\Omega_1, \Omega_2}$  on  $(\Delta_{\Omega_1, \Omega_2} \cap T) \setminus S$ . A routine identity argument shows that every two functions  $f_{\Omega_1, \Omega_2}$  coincide on the intersection of their domains of definition. Gluing  $f_{\Omega_1, \Omega_2}$ , we obtain the desired function  $\hat{f} \in \mathcal{O}(\bigcup \Delta_{\Omega_1, \Omega_2})$ .

**Step 2:  $N = 3$ .**

Consider the following elements of  $\Sigma_3$ .

$$\sigma_1 := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \sigma_2 := \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \sigma_3 := \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \sigma_4 := \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

Fix any subdomain  $\Omega_1 \times \Omega_2 \times \Omega_3 \subset D$  and pick any  $a_3 \in \mathcal{S}^{\sigma_1} \cap \mathcal{S}^{\sigma_2}$ . Then by the definition,  $\Omega_1 \setminus S^{\sigma_1}(a_3)$  (resp.  $\Omega_2 \setminus S^{\sigma_2}(a_3)$ ) is of Baire category I in  $\Omega_1$  (resp.  $\Omega_2$ ).

Also, for any  $a_1 \in S^{\sigma_1}(a_3) \cap \mathcal{S}^{\sigma_3}$ , we have  $\text{int } S(a_1, \cdot, a_3) = \emptyset$  and the set  $\Omega_2 \setminus \{a_2 \in \Omega_2 : \text{int } S(a_1, a_2, \cdot) = \emptyset\}$  is of Baire category I.

Similarly, for any  $a_2 \in S^{\sigma_2}(a_3) \cap \mathcal{S}^{\sigma_4}$ , we have  $\text{int } S(\cdot, a_2, a_3) = \emptyset$  and the set  $\Omega_1 \setminus \{a_1 \in \Omega_1 : \text{int } S(a_1, a_2, \cdot) = \emptyset\}$  is of Baire category I.

Thus  $f$  is well-defined on the union  $X$  of the two following subsets of  $\Omega_1 \times \Omega_2 \times \{a_3\}$ :

$$(5.2) \quad \{(z_1, z_2, a_3) : \text{ for any } z_1 \in S^{\sigma_1}(a_3) \cap \mathcal{S}^{\sigma_3}, \text{ and } z_2 \in S^{\sigma_3}(z_1) \cap S^{\sigma_2}(a_3)\}$$

and

$$(5.3) \quad \{(z_1, z_2, a_3) : \text{ for any } z_2 \in S^{\sigma_2}(a_3) \cap \mathcal{S}^{\sigma_4}, \text{ and } z_1 \in S^{\sigma_4}(z_2) \cap S^{\sigma_1}(a_3)\}.$$

Observe that by the definition in Lemma 4.1,  $\Omega_1 \setminus (S^{\sigma_4}(z_2) \cap S^{\sigma_1}(a_3))$  (resp.  $\Omega_2 \setminus (S^{\sigma_3}(z_1) \cap S^{\sigma_2}(a_3))$ ) is of Baire category I in  $\Omega_1$  (resp.  $\Omega_2$ ). By virtue of (5.2)–(5.3), the same conclusion also holds for the fibers  $X(z_1, \cdot, a_3)$  and  $X(\cdot, z_2, a_3)$ ,  $z_1 \in S^{\sigma_1}(a_3) \cap \mathcal{S}^{\sigma_3}$  (resp.  $z_2 \in S^{\sigma_2}(a_3) \cap \mathcal{S}^{\sigma_4}$ ).

Let  $\mathcal{U}_j \subset \Omega_j$  ( $j = 1, 2$ ) be an arbitrary open subset. If  $\Delta := \Delta_q(r)$  is a polydisc, then we denote by  $k\Delta$  the polydisc  $\Delta_q(kr)$  for all  $k > 0$ . Repeating the Baire category argument already used in the proof of Part (ii), one can show that there are a positive number  $K$ , polydiscs  $\Delta_j \subset \mathcal{U}_j$ , and subsets  $Q_{\mathcal{U}_1, \mathcal{U}_2}^1$  of  $S^{\sigma_1}(a_3) \cap \mathcal{S}^{\sigma_3}$

(resp.  $Q_{\mathcal{U}_1, \mathcal{U}_2}^2$  of  $S^{\sigma_2}(a_3) \cap \mathcal{S}^{\sigma_4}$ ) such that  $\overline{Q_{\mathcal{U}_1, \mathcal{U}_2}^j} = \Delta_j$ ,  $Q_{\mathcal{U}_1, \mathcal{U}_2}^j$  is of Baire category II, and  $\sup_{2\Delta_2} |f(\widetilde{a_1, \cdot, a_3})| \leq K$ , (resp.  $\sup_{2\Delta_1} |f(\widetilde{\cdot, a_2, a_3})| \leq K$ ), for  $a_j \in Q_{\mathcal{U}_1, \mathcal{U}_2}^j$ .

Therefore, by applying Lemma 4.2, we obtain a function  $f_{a_3} = f_{\mathcal{U}_1, \mathcal{U}_2, a_3} \in \mathcal{O}(\Delta_1 \times \Delta_2)$  which extends  $f(\cdot, \cdot, a_3)$  to  $\Delta_1 \times \Delta_2 \times \{a_3\}$  for all  $a_3 \in \mathcal{S}^{\sigma_1} \cap \mathcal{S}^{\sigma_2}$ .

Now let  $\mathcal{U}_j \subset \Omega_j$  ( $j = 1, 2, 3$ ) be an arbitrary open subset. Since the set  $\Omega_3 \setminus (\mathcal{S}^{\sigma_1} \cap \mathcal{S}^{\sigma_2})$  is of Baire category I, by using the previous discussion we are able to perform the Baire category argument already used in the proof of Part (ii). Consequently, there are a positive number  $K$ , polydiscs  $\Delta_j \subset \mathcal{U}_j$ , and subsets  $Q_{\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3}^3$  of  $\mathcal{S}^{\sigma_1} \cap \mathcal{S}^{\sigma_2}$  such that  $\overline{Q_{\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3}^3} = \Delta_3$ ,  $Q_{\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3}^3$  is of Baire category II, and  $\sup_{2\Delta_1 \times 2\Delta_2} |f_{a_3}(\cdot, \cdot)| \leq K$ , for  $a_3 \in Q_{\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3}^3$ .

By changing the role of 1, 2, 3 and by taking smaller polydiscs, we obtain in the same way the subsets  $Q_{\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3}^j \subset \Delta_j$  ( $j = 1, 2$ ) with similar property.

For  $j \in \{1, 2, 3\}$  consider the following subsets of  $T$

$$T_j := \{a = (a_1, a_2, a_3) : a_k \in \Delta_k, a_l \in \Delta_l, a_j \in Q_{\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3}^j, \{k, l, j\} = \{1, 2, 3\} \\ \text{and either } \text{int}_{\mathbb{C}^{n_l}} S(\cdot, a_k, a_j) = \emptyset \text{ or } \text{int}_{\mathbb{C}^{n_k}} S(a_l, \cdot, a_j) = \emptyset\}.$$

One next proves that

$$(5.4) \quad f(a) = f_{a_j}(a_k, a_l), \quad a \in T_1 \cup T_2 \cup T_3.$$

Indeed, let  $a = (a_1, a_2, a_3) \in T_3$  with  $\text{int } S(\cdot, a_2, a_3) = \emptyset$ . In virtue of (5.2), we can choose a sequence  $(z_1^n)_{n=1}^\infty \rightarrow a_1$  and for every  $n \geq 1$  a sequence  $(z_2^{m(n)})_{m=1}^\infty \rightarrow a_2$ . Clearly,  $f(z_1^n, z_2^{m(n)}, a_3) = f_{a_3}(z_1^n, z_2^{m(n)})$ . Therefore,

$$f(a) = \lim_{n \rightarrow \infty} f(z_1^n, a_2, a_3) = \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} f(z_1^n, z_2^{m(n)}, a_3) = f_{a_3}(a_1, a_2).$$

Now, we wish to glue the three functions  $f_{a_j}$  ( $j = 1, 2, 3$ ). Since the family  $\{f_{a_j} : a_j \in Q_{\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3}^j\}$  is normal, we define an extension  $f_j$  of  $f_{a_j}$  ( $j = 1, 2, 3$ ) to  $\Delta := \Delta_1 \times \Delta_2 \times \Delta_3$  as follows.

Let  $\{j, k, l\} \in \{1, 2, 3\}$  and for  $z = (z_1, z_2, z_3) \in \Delta$ , choose a sequence  $(a_j^n)_{n=1}^\infty \subset Q_{\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3}^j$  such that  $\lim_{n \rightarrow \infty} a_j^n = z_j$  and the sequence  $(f_{a_j^n})_{n=1}^\infty$  converges uniformly on compact subsets of  $\Delta_k \times \Delta_l$ . We let

$$(5.5) \quad f_j(z) := \lim_{n \rightarrow \infty} f_{a_j^n}(a_k^n, a_l^n),$$

for any sequence  $((a_k^n, a_l^n, a_j^n))_{n=1}^\infty \subset T_j \rightarrow z$  as  $n \rightarrow \infty$ .

Let us first check that the functions  $f_j$  are well-defined. Indeed, this assertion will follow from the estimate

$$(5.6) \quad |f_{a_j}(a_k, a_l) - f_{b_j}(b_k, b_l)| \leq CK|a - b|, \quad a = (a_k, a_l, a_j), b = (b_k, b_l, b_j) \in T_j.$$

Here  $C$  is a constant that depends only on  $\Delta$ . It now remain to prove (5.6) for example in the case  $j = 3$ . To do this, let  $z = (z_1, z_2, z_3), w = (w_1, w_2, w_3) \in T_3$ .

Then by virtue of (5.2) and (5.3), one can choose  $a_1, a_1' \in \Delta_1$  and  $a_2 \in \Delta_2$  such that

$$(5.7) \quad \begin{aligned} (a_1, a_2, z_3), (a_1', a_2, z_3) &\in T_2 \cap T_3, & |z - (a_1, a_2, z_3)| &\leq 2|z - w|, \\ |z - (a_1', a_2, z_3)| &\leq 2|z - w|. \end{aligned}$$

Write

$$\begin{aligned} |f_3(z) - f_3(w)| &\leq |f_3(z) - f_3(a_1, a_2, z_3)| + |f_3(a_1, a_2, z_3) - f_3(w)| \\ &\quad + \left| f_2(a_1, a_2, z_3) - f_2(a_1', a_2, z_3) \right|. \end{aligned}$$

Since  $\sup_{2\Delta_1 \times 2\Delta_2} |f_{z_3}| \leq K$ ,  $\sup_{2\Delta_1 \times 2\Delta_2} |f_{w_3}| \leq K$  and  $\sup_{2\Delta_1 \times \Delta_3} |f_{a_2}| \leq K$ , applying Schwarz's lemma to the right side of the latter estimate and using (5.7), the desired estimate (5.6) follows.

From the construction (5.5) above,  $f_j(\cdot, \cdot, z_j) \in \mathcal{O}(\Delta_k \times \Delta_l)$ . Moreover, a routine identity argument using (5.2) and (5.3) shows that  $f_1 = f_2 = f_3$ . Finally, define

$$\hat{f}_{\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3}(z) = f_1(z) = f_2(z) = f_3(z), \quad z \in \Delta,$$

then  $\hat{f}_{\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3}$  extends  $f$  holomorphically from  $T_1 \cup T_2 \cup T_3$  to  $\Delta$ . A routine identity argument as in (5.?) shows that  $\hat{f}_{\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3} = f$  on  $(T \cap \Delta) \setminus S$ . Gluing  $\hat{f}$  for all  $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3$ , we obtain the desired extension function  $\hat{f}$ . Hence the proof is complete in this case.

**Step 3:**  $N \geq 4$ .

The general case uses induction on  $N$ . Since the proof is very similar to the case  $N = 3$  making use of Lemmas 4.1 and 4.3 and using the inductive hypothesis for  $N - 1$ , we leave the details to the reader.  $\square$

## REFERENCES

- [1] M. Jarnicki, P. Pflug, *Extension of Holomorphic Functions*, de Gruyter Expositions in Mathematics **34**, Walter de Gruyter 2000.,
- [2] M. Jarnicki, P. Pflug, *An extension theorem for separately holomorphic functions with analytic singularities*, Ann. Pol. Math. (2002), to appear, 1–20.
- [3] M. Jarnicki, P. Pflug, *An extension theorem for separately holomorphic functions with pluripolar singularities*, Trans. Amer. Math. Soc. (2002), to appear, 1–19.
- [4] M. Jarnicki, P. Pflug, *An extension theorem for separately meromorphic functions with pluripolar singularities*, Kyushu J. of Math. (2003), to appear, arXiv:math.CV/0209207 v2, 1–11.
- [5] M. Klimek, *Pluripotential theory*, London Mathematical society monographs, Oxford Univ. Press., **6**, (1991).
- [6] Nguyễn Thanh Vân, J. Siciak, *Fonctions plurisousharmoniques extrémales et systèmes doublement orthogonaux de fonctions analytiques*, Bull. Sci. Math., **115**, (1991), 235–244.
- [7] Nguyễn Thanh Vân, A. Zeriahi, *Une extension du théorème de Hartogs sur les fonctions séparément analytiques*, Analyse Complexe Multivariables, Récents Développement, A. Meril (ed.), EditEl, Rende, (1991), 183–194.
- [8] W. Rothstein, *Ein neuer Beweis des Hartogsschen Hauptsatzes und seine Ausdehnung auf meromorphe Funktionen*, Math. Z., **53**, (1950), 84–95.
- [9] E. Sakai, *A note on meromorphic functions in several complex variables*, Memoirs of the Faculty of Science, Kyusyu Univ., **11**, (1957), 75–80.

- [10] Y. T. Siu, *Techniques of Extension of Analytic Objects*, Lectures Notes in Pure and Appl. Math., **8**, Marcel Dekker, (1974).

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